

The Bernstein-Sato b-function for the complement of the open SL_n -orbit on a triple flag variety

Henry Scher

July 1, 2015

Abstract

We calculate Bernstein-Sato b-functions for $f_{G^3}^\lambda$, a SL_n -invariant section of a line bundle on $SL_n/B \times SL_n/B \times \mathbb{P}^{n-1}$ whose zero-set is the complement of the open G -diagonal orbit. The proof uses a similar calculation by Kashiwara of the b-function for f^λ , a B^- -semiinvariant section of a line bundle on SL_n/B whose zero-set is the complement of the big Bruhat cell.

1 Introduction

Let X be an algebraic variety over \mathbb{C} , and let f be an algebraic function on X . The Bernstein-Sato function of f is defined by the monic generator of the ideal of $b(s)$ such that there is a differential operator $P(s)$ such that

$$P(s)f^{s+1} = b(s)f^s \quad (1)$$

where $P(s) \in \mathcal{D}(X)[s]$. There have been various generalizations to multiple functions. In this paper, the ideal will be defined by the following functional equation:

$$P_{m_1, m_2, \dots, m_n}(s_1, s_2, \dots, s_n) \prod_i f_i^{s_i + m_i} = b_{m_1, m_2, \dots, m_n}(s_1, s_2, \dots, s_n) \quad (2)$$

where $P(s) \in \mathcal{D}(X)[s_1, s_2, \dots, s_n]$, $m_i \in \mathbb{Z}$. Sabbah [4] and Gyoja [5] generalized Kashiwara's result about rationality of the roots of b-functions by showing that for each choice of $\{t_i\}$, there is an element in the b-function ideal that can be written as the product of hyperplanes (i.e. functions of the form $\sum_i a_i t_i + k_i$).

Let $G = SL_n$, B a Borel subgroup, and \mathcal{F} be the corresponding flag variety. The action of G on \mathcal{F} has a unique open B -orbit; its complement is a hypersurface, so there are B -semiinvariant global sections f^λ of G -equivariant line

bundles (which, for each line bundle, are unique when they exist). The zero-sets of these sections are components of the hypersurface. In 1985, Kashiwara [1] found the b-function for these sections using universal Verma modules.

Let $X = \mathcal{F} \times \mathcal{F} \times \mathbb{P}^{n-1}$. Then the diagonal G -action on X has an open orbit with a hypersurface complement; correspondingly, there are G -semiinvariant global sections (for a specific character) of G -equivariant line bundles \mathcal{L} on X whose zero-sets are components of the complement of this open orbit [2]. It has been conjectured that the b-function of these sections could be found using techniques similar to those used by Kashiwara. This paper finds those b-functions.

We start in section 3 by defining the G -invariant global sections f_G^λ of line bundles on X , and the relevant functional equation for the type of b-functions we wish to use. In section 4, we explain some of Kashiwara's techniques, including finding differential operators on \mathcal{F} , and use his results to find a relationship among his differential operators. In section 5, we use Kashiwara's differential operator to find the differential operator on X that satisfies the functional equation, and a relationship between the b-function Kashiwara found for a different global section and the b-function for the global section examined in this paper, through a function $H(\lambda)$; we also show that there is a similar relationship between the differential operators for the G -invariant global sections. In section 6, we examine $H(\lambda)$ more closely and find what it is for some of the G -invariant global sections. In section 7, we use the relationships between the differential operators in order to prove one of the conjectures from Ginzburg and Finkelberg [2], that the b-functions can be factored into linear factors, and also prove some relationships between the b-functions. In section 8, we use the values given in section 6 with the relationships in section 5 to fully determine all of the b-functions.

2 Notation

1. $G = SL_n$
2. B is a Borel subgroup of G .
3. P is a mirabolic subgroup of G , that is, a subgroup of G that fixes a line in \mathbb{C}^n .
4. $\mathcal{F} = G/B$ is the flag variety of G , thought of as the moduli space of Borel subgroups.
5. $X = \mathcal{F} \times \mathcal{F} \times \mathbb{P}^{n-1}$ is the moduli space of triples of two Borel subgroups and a mirabolic subgroup.
6. V_{λ_1} is the irreducible representation of G with highest weight λ_1 .
7. If $\lambda = (\lambda_1, \lambda_2, l)$, then $V_\lambda := V_{\lambda_1} \otimes V_{\lambda_2} \otimes \text{Sym}^l \mathbb{C}^n$ is an irreducible representation of $G^3 = G \times G \times G$.

8. $B_1 \times B_2 \times P$ is a triple of subgroups of G , with B_1, B_2 Borel and P mirabolic.
9. $B'_1 \times B'_2 \times P'$ is a triple of subgroups of G , with B'_1, B'_2 Borel and P' dual-mirabolic (i.e. fixing a line in \mathbb{C}^n with the dual action of G).
10. v_λ is a highest weight vector in V_λ with respect to $B_1 \times B_2 \times P$.
11. $v_{-\lambda}$ is a highest weight vector in V_λ^* with respect to $B'_1 \times B'_2 \times P'$ such that $\langle v_{-\lambda}, v_\lambda \rangle = 1$.
12. Ω is the set of triples λ such that there is a diagonally G -invariant vector $u_{-\lambda} \in V_\lambda^*$ with $\langle u_{-\lambda}, v_\lambda \rangle = 1$.
13. $u_\lambda \in V_\lambda$ is the diagonally G -invariant vector with $\langle v_{-\lambda}, u_\lambda \rangle = 1$.
14. $H(\lambda) = \frac{\langle v_{-\lambda}, v_\lambda \rangle \langle u_{-\lambda}, u_\lambda \rangle}{\langle v_{-\lambda}, u_\lambda \rangle \langle u_{-\lambda}, v_\lambda \rangle}$
15. ω is the standard representation \mathbb{C}^n of G
16. $\wedge^i \omega_1$ is the i th fundamental representation of SL_n .
17. ω_j is the standard representation of $SL_j \subset G$.

3 The relevant line bundle and section

Choose two Borel subgroups B_1, B_2 and a mirabolic subgroup P of $G = SL_n$, i.e. a subgroup P is the stabilizer of a point in \mathbb{P}^{n-1} ; such a subgroup is called a mirabolic subgroup. This subgroup is the stabilizer of a unique point $x \in X$, so X can be seen as the moduli space of such triples. We further require that they be in general position; that is, that x lie in the open G -diagonal orbit. Let Γ be the lattice of triples $(\lambda_1, \lambda_2, l)$ of pairs of integral weights and an integer; let $\Gamma_{\geq 0}$ be the subcone where the weights are dominant and the integer is nonnegative.

For $\lambda = (\lambda_1, \lambda_2, l) \in \Gamma_{\geq 0}$, define $V_\lambda = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \text{Sym}^l \mathbb{C}^n$. Let Ω be the set of $\lambda \in \Gamma_{\geq 0}$ such that V_λ^* has a nontrivial G -invariant element.

Lemma 1. Ω is a cone (i.e. it is closed under addition).

Proof. By the Borel-Weil theorem, the global sections of the line bundle L_λ on $X = \mathcal{F} \times \mathcal{F} \times \mathbb{P}^{n-1}$ form a representation of G^3 isomorphic to V_λ^* . As such, V_λ^* contains a nontrivial G -invariant element under the diagonal action of G if and only if there is a nontrivial global G -invariant section of L_λ . Then because X is an irreducible variety (and as such has a homogeneous coordinate ring without zero divisors), if there is a nontrivial G -invariant element of V_λ^* and another of $V_{\lambda'}^*$, then there necessarily is a nontrivial G -invariant element of $V_{\lambda+\lambda'}^*$ corresponding to the product of the two invariant sections in $L_{\lambda+\lambda'}$. \square

For $\lambda \in \Gamma_{\geq 0}$, define $v_\lambda \in V_\lambda$ as a nonzero highest-weight element with respect to $B_1 \times B_2 \times P$. Because we have chosen our subgroups in general position, $v_\gamma^G \neq 0$ for all $\gamma \in \Gamma_{\geq 0}$.

Define $X' = \mathcal{F} \times \mathcal{F} \times (\mathbb{P}^{n-1})^\vee$ with a G^3 -action, where the G -action on $(\mathbb{P}^{n-1})^\vee$ comes from the action of G on the dual of the usual representation. Then X' corresponds to the moduli space of triples of two Borel subgroups B'_1, B'_2 and a dual-mirabolic subgroup P' (i.e. a mirabolic subgroup fixing an element of the dual of the standard representation). We then say that $x' \in X'$ is in general position if it is in the open G -orbit on X' . If we choose an invariant element $u_{-\lambda} \in V_\lambda^*$, then $\langle u_{-\lambda}, v_\lambda \rangle = 0$ is the identifying equation of a closed subvariety of the complement of the open orbit. Therefore x' being in the open orbit is equivalent to the existence for any $\lambda \in \Omega$ of a $u_{-\lambda} \in V_\lambda^*$ that is G -invariant with $\langle u_{-\lambda}, v_\lambda \rangle = 1$ (by scaling). Finally, a point $(x, x') \in X \times X'$ is in general position if both x and x' are in general position and each subgroup and its primed counterpart are opposite. Equivalently (for similar reasoning to the above), for any $\lambda \in \Gamma_{\geq 0}$, we want there to be an element $v_{-\lambda} \in V_\lambda^*$ of highest weight with respect to $B'_1 \times B'_2 \times P'$ such that $\langle v_{-\lambda}, v_\lambda \rangle = 1$. Borel-Weil implies that the correspondence between V_λ^* and $\Gamma(X, L_\lambda)$ can be given by $w \rightarrow [g \rightarrow \langle w, gv_\lambda \rangle]$. We therefore have two sections of L_λ , $f_{G^3}^\lambda$ and f_K^λ , such that $f_{G^3}^\lambda$ is diagonally G -invariant, while f_K^λ is left $B'_1 \times B'_2 \times P'$ semiinvariant. The former subscript denotes that the section is G -invariant; the latter denotes that the section comes from the work of Kashiwara.

Because there is both a unique open G -orbit and a unique open $B'_1 \times B'_2 \times P'$ -orbit on X , these sections are the unique ones with this property up to scaling. Then as products of semiinvariants are semiinvariant, and by evaluating at the identity on G^3 , we can see that $f_K^\lambda f_K^\mu = f_K^{\lambda+\mu}$ and $f_{G^3}^\lambda f_{G^3}^\mu = f_{G^3}^{\lambda+\mu}$.

The b-function we want to find is that of $f_{G^3}^\lambda$; to be more precise, we want to find solutions to the following functional equation:

$$P_\mu f_{G^3}^{\lambda+\mu} = b_\mu(\lambda) f_{G^3}^\lambda \quad (3)$$

where P_μ is a twisted differential operator on \mathcal{F} (equivalently, a differential operator on G such that for any right B -semiinvariant f , $P_\mu f$ is also right B -semiinvariant).

This corresponds to a b-function using the definition given by Gyoja [5], where the f_i are f_λ for λ a generator of Ω . We will describe those generators in section 5.

4 Kashiwara's argument and the differential operator

Let G be any semisimple complex simply connected Lie group with Lie algebra \mathfrak{g} ; let B be a Borel subgroup and B' be an opposite Borel subgroup with common torus $H = B \cap B'$. Then Kashiwara wanted to find the b-function of the B' -semiinvariant section f_G^λ of the line bundle L_λ over $\mathcal{F} = G/B$, using its

cover (which we will also call f_G^λ through abuse of notation) $f_G^\lambda : G \rightarrow \mathbb{C}, g \rightarrow \langle v_{-\lambda}, gv_\lambda \rangle$ where λ is an integral dominant weight, $v_\lambda \in V_\lambda$ is a highest weight vector wrt B (and therefore also with respect to H), and $v_{-\lambda} \in V_\lambda^*$ is a highest weight vector wrt B' (and lowest weight wrt H).

Let R_+ be the set of positive roots of $G = SL_n$ with respect to B , $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$, and for any $\alpha \in R_+$, let $h_\alpha(\mu) = \langle \alpha, \mu \rangle$.

Define $\mathcal{D}^\mu(\mathcal{F})$ as the set of differential operators that twist by $-\mu$. In other words, if $P \in \mathcal{D}^\mu(\mathcal{F})$ and $f \in \mathcal{L}_\lambda$, then $Pf \in \mathcal{L}_{\lambda-\mu}$.

Theorem 2. (Kashiwara) [1] *There is a twisted differential operator $P_\mu \in \mathcal{D}^\mu(\mathcal{F})$ on \mathcal{F} such that for any λ, μ nonnegative weights for G :*

$$P_\mu f_G^{\lambda+\mu} = b_\mu(\lambda) f_G^\lambda \quad (4)$$

$$\text{where } b_\mu(\lambda) = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha(\mu)} (h_\alpha(\lambda) + h_\alpha(\rho) + i).$$

Proof. Following Kashiwara, we can trivialize the sheaf of differential operators on G using right translation: $\mathcal{D}(G) = \mathbb{C}[G] \otimes R(U\mathfrak{g})$, where $R : U\mathfrak{g} \rightarrow \mathcal{D}(G)$ is the extension to $U\mathfrak{g}$ of the map $R : \mathfrak{g} \rightarrow TG$ given by right translation.

Then by quantum Hamiltonian reduction [3], $\mathcal{D}^\mu(\mathcal{F}) = \left(\frac{\mathcal{D}(G)}{\mathcal{D}(G)\mathfrak{n}} \right)^{\mathfrak{b}-\mu}$, where the subscript on \mathfrak{b} denotes the subset of elements of weight $-\mu$. The quotient $\frac{\mathcal{D}(G)}{\mathcal{D}(G)\mathfrak{n}} \simeq \mathbb{C}[G] \otimes (U\mathfrak{g}/((U\mathfrak{g})\mathfrak{n}))$. Using the well-known decomposition $\mathbb{C}[G] = \oplus (V_\nu \otimes V_\nu^*)$ as G -bimodules and rearranging tensor products, we get that $\mathcal{D}_\mu(\mathcal{F}) = \oplus (V_\nu \otimes (V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}))^{\mathfrak{b}-\mu}$. Because we trivialized by right translation, and because the decomposition separates the actions of left and right translation, $\mathcal{D}^\mu(\mathcal{F}) = \oplus V_\nu \otimes (V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$, where G acts on V_ν , while

We will use two results that Kashiwara proved; the proofs are given in the appendix. Define $U\mathfrak{h} = S\mathfrak{h}$ as the symmetric algebra and universal enveloping algebra of \mathfrak{h} .

Theorem 3. *Let V be a finite dimensional \mathfrak{b} -module. Then $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ is a $S\mathfrak{h}$ module of rank equal to the dimension of the μ weight subspace of V .*

Theorem 4. *Assume that the dimension of V^μ is 1. Then $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ is free as a right $U\mathfrak{h}$ -module.*

The first theorem implies that if $\nu \not\geq \mu$, then $(V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ is trivial. The second then shows that in the minimal case of $\nu = \mu$, $(V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ is a free

rank 1 right $U\mathfrak{h}$ -module. Call the generating element $P \in (V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$.

Then by attaching a highest weight vector $v_\mu \in V_\mu$, we get a differential operator $P_\mu = v_\mu \otimes P \in \mathcal{D}_\mu(\mathcal{F})$. Note that P_μ varies under left translation as v_μ does, that is, $gP_\mu = (gv_\mu) \otimes P$ where g acts on differential operators by left translation.

By the definition of $\mathcal{D}_\mu(\mathcal{F})$, $P_\mu f^{\lambda+\mu} \in \Gamma(\mathcal{F}, L_\lambda)$. Further, as the map of application of differential operators $\mathcal{D}_\mu(\mathcal{F}) \otimes \Gamma(\mathcal{F}, L_{\lambda+\mu}) \rightarrow \Gamma(\mathcal{F}, L_\lambda)$ is G -equivariant, the image will be of weight $-\lambda$. By Borel-Weil, $\Gamma(\mathcal{F}, L_\lambda) \simeq V_\lambda^*$, so there is only one element of weight $-\lambda$ (up to scaling). But f_λ is already of weight $-\lambda$ - so we get that $P_\mu f^{\lambda+\mu} = b_\mu(\lambda) f^\lambda$.

Kashiwara further found an explicit formula:

$$b_\mu(\lambda) = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha(\mu)} (h_\alpha(\lambda) + h_\alpha(\rho) + i)$$

and that this is the generator of the ideal of all solutions to the functional equation. \square

One result was not stated by Kashiwara but which will be useful for us is the following:

Lemma 5. *For any μ, ν , $P_\mu P_\nu = P_{\mu+\nu}$.*

Proof. Consider the decomposition of $\mathbb{C}[G]$ into $\oplus V_\mu \otimes V_\mu^*$ as an increasing algebra filtration, $\mathbb{C}[G]_{\leq \mu} = \oplus_{\nu \leq \mu} V_\nu \otimes V_\nu^*$ with $\mathbb{C}[G]_{\leq \mu} \mathbb{C}[G]_{\leq \nu} \subseteq \mathbb{C}[G]_{\leq \mu+\nu}$, where $\mu \leq \nu$ is the usual partial order on weights. Then by using the trivial filtration on $U\mathfrak{g}$, we obtain an increasing filtration on $\mathcal{D}(G) \simeq \mathbb{C}[G] \otimes U\mathfrak{g}$; as the action of \mathfrak{g} respects the filtration on $\mathbb{C}[G]$, this filtration is an increasing algebra filtration. This filtration then descends to $\mathcal{D}(\mathcal{F})$, so we obtain an increasing filtration $\mathcal{D}_\mu^{\leq \eta}(\mathcal{F}) = \oplus_{\nu \leq \eta} V_\nu \otimes (V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$.

The highest weight in $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ is 0, so if $\nu < \mu$, then $\mathcal{D}_\mu^{\leq \nu} = \{0\}$, and $\mathcal{D}_\mu^{\leq \mu}(\mathcal{F}) = V_\mu \otimes (V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$. By definition, $P_\mu \in \mathcal{D}_\mu^{\leq \mu}(\mathcal{F})$. Therefore, $P_\mu P_\nu \in \mathcal{D}_{\mu+\nu}^{\leq (\mu+\nu)}(\mathcal{F})$.

Under left translation, P_μ acts as v_μ , and therefore has weight μ . Therefore, $P_\mu P_\nu$ has weight $\mu + \nu$. But $P_\mu P_\nu \in \mathcal{D}_{\mu+\nu}^{\leq \mu+\nu}(\mathcal{F}) = V_{\mu+\nu} \otimes (V_{\mu+\nu}^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu-\nu}$, where left translation acts on the $V_{\mu+\nu}$. As $v_{\mu+\nu} \in V_{\mu+\nu}$ is the unique element of weight $\mu + \nu$ up to scaling, $P_\mu P_\nu$ can be expressed as $v_{\mu+\nu} \otimes P'$ for some $P' \in (V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$; similarly, $P_{\mu+\nu} = v_{\mu+\nu} \otimes P$. Then by Kashiwara's proof

that P generates $(V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$ over $U\mathfrak{h}$, $P' = Pa$ for some $a \in U\mathfrak{h}$. Using

Kashiwara's calculation, we can check that $P_{\mu+\nu} f^{\lambda+\mu+\nu} = P_\mu P_\nu f^{\lambda+\mu+\nu}$ for any λ , which means that $a = 1$ and $P' = P$ - so $P_\mu P_\nu = P_{\mu+\nu}$. \square

5 Relationships among P_μ

We have proven that $P_\mu P_\nu = P_{\mu+\nu}$. This situation is common enough that it is useful to study it on its own; it will turn out to be true for the two relevant families of differential operators studied in this paper. This equation on differential operators implies the following equation on b-functions (with subscripts omitted because it works for either subscript):

$$b_\mu(\lambda)b_\nu(\lambda + \mu) = b_{\mu+\nu}(\lambda) \quad (5)$$

which can be seen as a 1-cocycle equation in the bar resolution complex for the group cohomology of Λ , where the action of Λ is on the multiplicative group $\mathbb{C}(\mathbb{C}\Lambda)^\times$ of rational functions (not including 0) on $\mathbb{C}\Lambda$ by translation, $T_\mu(p)(\lambda) = p(\lambda + \mu)$.

Lemma 6. *Let $\{b_\mu\}$ be a cocycle (i.e. satisfy the above equation) such that for $\mu \in \Omega$, b_μ is a polynomial (not just a rational function). Then for any $\mu \in \Omega$, b_μ can be expressed as $\prod_i (\alpha_i(\lambda) + k_i)$ with $\alpha_i \in \text{Hom}(\Lambda, \mathbb{Z})$, $k_i \in \mathbb{C}$.*

Similar results have been found for b-functions of multiple functions; see Sabbah, theorem 4.2.1 [4] and Gyoja [5].

Proof. From the cocycle equation, we can switch μ, ν to get that $b_\mu T_\mu(b_\nu) = b_\nu T_\nu(b_\mu)$.

Assume $p|b_\mu$ for some irreducible p ; we need to prove that p is of the form $\alpha(\lambda) + k$ for some $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$, $k \in \mathbb{C}$. We first check that for any $\nu \in \Omega$, either there is a nonnegative integer a such that $p|T_{\nu-a\mu}b_\mu$ or p is invariant under translation by μ .

By the second equation, we know that $p|b_\nu T_\nu b_\mu$. As polynomials over $\mathbb{C}\Lambda$ form a UFD and p is irreducible, we know that either $p|b_\nu$ or $p|T_\nu b_\mu$. In the latter case, we are done, so assume we are in the former case. Then by translation, we also have that $T_\mu p|T_\mu b_\nu|b_\mu T_\mu b_\nu = b_\nu T_\nu b_\mu$. By repeating this, we either have infinitely many a such that $p(\lambda + a\mu)|b_\nu(\lambda)$, or for some a that $T_{a\mu}p|T_\nu b_\mu$. As before, assume we are in the former case. Then as $b_\nu(\lambda)$ has finitely many factors, there can only be finitely many distinct $T_{a\mu}p$; therefore they must be equal for some a_0, a_1 , and therefore, $T_{(a_1-a_0)\mu}p = p$. This means that for any λ , the function $t \rightarrow p(\lambda + t\mu)$ is periodic. But p is a polynomial, so the function is a polynomial, which can only be periodic if it is constant, so $p(\lambda + t\mu) = p(\lambda)$ for any t , and p is invariant under translation by any multiple of μ .

Assume we are in the first case, that is, that $T_{a\mu}p|T_\nu b_\mu$. Then we know that $T_{a\mu-\nu}p|b_\mu$. As the former is a translation of p , it is also irreducible - so we can repeat the above process. As b_μ has finitely many factors, there can only be finitely many distinct such translations; therefore, for some $m_0 < m_1, n_0 < n_1$ we get that $T_{m_1\mu-n_1\nu}p = T_{m_0\mu-n_0\nu}p$. Therefore there are integers m, n with $n > 0$ such that $T_{m\mu-n\nu}p = p$. Dropping the assumption that $T_{a\mu}p|T_\nu b_\mu$ and going back to the general case (i.e. allowing $T_{a\mu}p = p$ as in the above

paragraph), we get that there are integers m, n nonnegative and not both 0 such that $T_{m\mu-n\nu}p = p$.

Choose generators β_i of Λ ; then as p is a nonconstant polynomial, for some i_0 we get that $T_{\beta_{i_0}}p \neq p$. Then for all $i \neq i_0$, we know that there are m_i, n_i such that $p(\lambda + m_i\beta_{i_0} - n_i\beta_i) = p(\lambda)$. Hence $n_i \neq 0$. For any $\mu \in \text{span}(\{m_i\beta_{i_0} - n_i\beta_i\})$, we have that $p(\lambda + \mu) = p(\lambda)$. This is a subspace of codimension 1 in $\mathbb{C}\Lambda$, so there is some $\alpha \in \text{Hom}(\Lambda, \mathbb{C})$ such that $\alpha(\mu) = 0$ if and only if $p(\lambda + \mu) = p(\lambda)$ for all λ ; further, it's clear that $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$. We then have that for any μ such that $\langle \alpha, \mu \rangle = 0$, $p(\lambda + \mu) = p(\lambda)$. Choose a point λ_0 such that $p(\lambda_0) = 0$. Then for any point where $\alpha(\lambda) - \alpha(\lambda_0) = 0$, we have that $p(\lambda) = p(\lambda + (\lambda_0 - \lambda)) = 0$. But $\alpha(\lambda) - \alpha(\lambda_0)$ is an irreducible polynomial, so $\alpha(\lambda) - \alpha(\lambda_0)$ divides p . Since p is irreducible, $p = \alpha(\lambda) - \alpha(\lambda_0)$. Setting $k = -\alpha(\lambda_0)$, we have proven the lemma. \square

Remark. A useful way to think about this result is to consider the bar resolution complex in which the b-function is a 1-cocycle. A generalization of this proof can be used to give the first cohomology group of the complex. We can then provide a "lift" $A(\lambda)$ for each cocycle in a canonical way as a product of gamma functions of rational hyperplanes, such that $b_\mu(\lambda) = \frac{A(\lambda + \mu)}{A(\lambda)}$, $A(0) =$

1. For any $\mu \in \Lambda$, we then have that $b_\mu(0) = \frac{A(\mu)}{A(0)} = A(\mu)$. The following corollaries are then natural.

Consider $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$ positive and indivisible (that is, there is no non-trivial $a \in \mathbb{Z}$ with $\frac{\alpha}{a} \in \text{Hom}(\Lambda, \mathbb{Z})$, and for any $\mu \geq 0$, $\langle \alpha, \mu \rangle \geq 0$). For each $\mu \in \Lambda^+$, let $K_{\alpha, \mu} = \{ks.t. (\alpha(\mu) + k)|b_\mu\}$ with multiplicity. Then for any $\mu, \nu \in \Lambda^+$

$$K_{\alpha, \mu} \cup (K_{\alpha, \nu} + \langle \alpha, \mu \rangle) = K_{\alpha, \nu} \cup (K_{\alpha, \mu} + \langle \alpha, \nu \rangle) \quad (6)$$

where $A + k = \{a + k | a \in A\}$; this follows directly from unique factorization and the cocycle equation.

Corollary 7. *Let $\alpha \neq 0 \in \text{Hom}(\Lambda, \mathbb{Z})$. Then there is some $c \in \mathbb{Z}_{\geq 0}$ such that $|K_{\alpha, \mu}| = c\langle \alpha, \mu \rangle$.*

Proof. Let $\nu \in \Lambda^+$, $\langle \alpha, \nu \rangle \neq 0$, and let $c = \frac{|K_{\alpha, \nu}|}{\langle \alpha, \nu \rangle}$. Then for any $\mu \in \Lambda^+$, by summing the K equation over the sets with multiplicity, we get that:

$$\sum K_{\alpha, \mu} + \sum K_{\alpha, \nu} + |K_{\alpha, \nu}|\langle \alpha, \mu \rangle = \sum K_{\alpha, \nu} + \sum K_{\alpha, \mu} + |K_{\alpha, \mu}|\langle \alpha, \nu \rangle. \quad (7)$$

We can subtract and divide to get $c = \frac{|K_{\alpha, \nu}|}{\langle \alpha, \nu \rangle}$; then as α is indivisible and positive, c must be a nonnegative integer. \square

Corollary 8. For any $\alpha \neq 0 \in \text{Hom}(\Lambda, \mathbb{Z})$, $\mu \in \Lambda$ with $\langle \alpha, \mu \rangle = 0$, $K_{\alpha, \mu} = \emptyset$.

Corollary 9. Let $\mu, \nu \in \Lambda^+$, and let $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$ such that $\alpha(\mu) = \alpha(\nu) = a$. Then $K_{\alpha, \mu} = K_{\alpha, \nu}$.

Proof. Assume otherwise. If we allow

$$(K_{\alpha, \mu} + a) - K_{\alpha, \mu} = (K_{\alpha, \nu} + a) - K_{\alpha, \nu}$$

But the map $S \rightarrow (S + a) - S$ is a \mathbb{Z} -linear map where every element of the kernel has infinitely many elements, and both $K_{\alpha, \mu}$ and $K_{\alpha, \nu}$ are finite - so they must be the same. \square

6 The differential operator for f_G

We now want to find the differential operators P_{μ, G^3} that give us the functional equation

$$P_{\mu, G^3} f_G^{\lambda + \mu} = b_{\mu, G^3}(\lambda) f_G^\lambda \quad (8)$$

In order to do this, we look first for the b-function for f_K , the $B_1 \times B_2 \times P$ -semiinvariant section, using Kashiwara's calculation, and then figure out the relationship between b_{μ, G^3} and $b_{\mu, K}$. As $f_K = f_G \otimes f_G \otimes f_{\mathbb{P}}$ is a product of three functions on the three factors of X , we can set $P_{\mu, K} = P_{G, \mu_1} \otimes P_{G, \mu_2} \otimes P_{\mathbb{P}, m}$ and get the functional equation

$$\begin{aligned} P_{\mu, K} f_K^{\lambda + \mu} &= (P_{G, \mu_1} f_G^{\lambda_1 + \mu_1}) \otimes (P_{G, \mu_2} f_G^{\lambda_2 + \mu_2}) \otimes (P_{\mathbb{P}, m} f_{\mathbb{P}}^{l + m}) \\ &= b_{G, \mu_1}(\lambda_1) b_{G, \mu_2}(\lambda_2) b_{\mathbb{P}, m}(l) f_K^\lambda \end{aligned}$$

where $b_{\mathbb{P}, m}$ is the b-function for a hyperplane in \mathbb{P}^{n-1} . It is easy to see that $b_{\mathbb{P}, m}(l) = \prod_{i=1}^m (l + i)$; combining that with the calculation in section 5, we get that:

$$P_{\mu, K} f_K^{\lambda + \mu} = b_{\mu, K}(\lambda) f_K^\lambda \quad (9)$$

where

$$b_{\mu, K} = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha \mu_1} (h_\alpha(\lambda_1) + h_\alpha(\rho) + i) \prod_{j=1}^{h_\alpha \mu_2} (h_\alpha(\lambda_2) + h_\alpha(\rho) + j) \prod_{k=1}^m (l + i)$$

As we now have $b_{\mu, K}$, we only need to determine the relationship between the two b-functions.

Lemma 10. Define $H(\mu) = \langle u_{-\mu}, u_\mu \rangle$. Then $u_\mu = H(\mu) v_\mu^G$, $u_{-\mu} = H(\mu) v_{-\mu}^G$.

Proof. Because u_μ is the unique G -invariant up to scaling, $u_\mu = c v_\mu^G$ for some c . Then $H(\mu) = \langle u_{-\mu}, u_\mu \rangle = \langle u_{-\mu}, c v_\mu^G \rangle = c \langle u_{-\mu}, v_\mu \rangle = c$, so $u_\mu = H(\mu) v_\mu^G$. Similarly, $u_{-\mu} = H(\mu) v_{-\mu}^G$. \square

Let $P_{\mu,G} = u_\mu \otimes P \in \mathcal{D}_\mu(X)$. Then as $P_{\mu,K} = v_\mu \otimes P$, the last lemma shows that $P_{\mu,G^3} = H(\mu)P_{\mu,K}^G$.

Lemma 11. $P_{\mu,G^3} f_{G^3}^{\lambda+\mu} = b_{\mu,G^3}(\lambda) f_{G^3}^\lambda$ where $b_{\mu,G^3}(\lambda) = \frac{H(\lambda+\mu)}{H(\lambda)} b_{\mu,K}$

Proof. Consider differential operators of the form $V_\mu \otimes P$, and the restriction of the map of application of differential operators $\mathcal{D}_\mu(X) \otimes \Gamma(X, L_{\lambda+\mu}) \rightarrow \Gamma(X, L_\lambda)$ to only include such differential operators. Then this is isomorphic to a $G \times G \times G$ -equivariant map $V_\mu \otimes V_{\lambda+\mu}^* \rightarrow V_\lambda^*$, where the isomorphism takes $P_{\mu,K} \rightarrow v_\mu, P_{\mu,G} \rightarrow u_\mu, f_K^\lambda \rightarrow v_{-\lambda}, f_G^\lambda \rightarrow u_{-\lambda}$. But up to scaling, there is only one such map. Under this interpretation, we have $v_\mu \otimes v_{-\lambda-\mu} \rightarrow b_{K,\mu}(\lambda) v_{-\lambda}$ and $u_\mu \otimes u_{-\lambda-\mu} \rightarrow b_{G,\mu}(\lambda) u_{-\lambda}$. By pairing with the respective vectors and considering the fact that this map is $G \times G \times G$ -invariant, we can think of this as a map $V_\mu \otimes V_\lambda \rightarrow V_{\mu+\lambda}$ that takes $v_\mu \otimes v_\lambda \rightarrow v_{\mu+\lambda}, H(\lambda+\mu) u_\mu \otimes u_\lambda \rightarrow H(\lambda) b_{G,\mu}(\lambda) u_{\lambda+\mu}$.

But by Borel-Weil, we also have an isomorphism between $\Gamma(X', L_{-\lambda})$ and V_λ ; as such, we have the multiplication map on sections. The multiplication map takes $v_\mu \otimes v_\lambda$ to $v_{\mu+\lambda}$ - so the map we want is the multiplication map scaled by $b_{K,\mu}$. The multiplication map also takes $u_\mu \otimes u_\lambda$ to $u_{\mu+\lambda}$. Therefore, the map we want takes $H(\lambda+\mu) u_\mu \otimes u_\lambda$ to $H(\lambda+\mu) u_{\lambda+\mu}$. So $b_{\mu,G^3}(\lambda) = \frac{H(\lambda+\mu)}{H(\lambda)} b_{\mu,K}$. \square

Remark. This lemma is easier to understand in terms of the lift of the cocycle, as explained in the remark in the previous section. If $A_K(\lambda)$ is the lift of $b_{\mu,K}$ and $A_{G^3}(\lambda)$ is the lift of b_{μ,G^3} , then for $\lambda \in \Omega$, $A_{G^3}(\lambda) = H(\lambda) A_K(\lambda)$.

Similar reasoning applied to the map $V_\mu \rightarrow \mathcal{D}_\mu(X), v \rightarrow v \otimes P$ and the differential operator multiplication map also implies that $P_{\mu,G^3} P_{\nu,G^3} = P_{\mu+\nu,G^3}$. As such, all of the conclusions from the previous section apply to the b-function we are trying to find.

7 The functions $H(\lambda)$

We now wish to find $H(\lambda)$. The value of $H(\lambda)$ depends on the choice of the 6 subgroups from before; for certain values of λ , we can choose convenient subgroups to make the calculation easier. As such, we first need to show how the choice of subgroup changes $H(\lambda)$.

In the last section, we defined $H(\lambda) = \langle u_{-\lambda}, u_\lambda \rangle$ where u_λ and $u_{-\lambda}$ were normalized with $\langle u_{-\lambda}, v_\lambda \rangle = \langle v_{-\lambda}, u_\lambda \rangle = \langle v_{-\lambda}, v_\lambda \rangle = 1$; however, $H(\lambda)$ is easier to understand in an "un-normalized" form, $H(\lambda) = \frac{\langle u_{-\lambda}, u_\lambda \rangle \langle v_{-\lambda}, v_\lambda \rangle}{\langle u_{-\lambda}, v_\lambda \rangle \langle v_{-\lambda}, u_\lambda \rangle}$. In this form, we can instead choose arbitrary G -invariant $u_\lambda, u_{-\lambda}$, $B_1 \times B_2 \times P$ -semiinvariant v_λ , and $B'_1 \times B'_2 \times P'$ -semiinvariant $v_{-\lambda}$. We can then regard $H(\lambda)$ as a meromorphic function $H_\lambda(x \times x')$ on $X \times X'$.

Lemma 12. The function $\frac{H_{\lambda+\mu}}{H_\lambda H_\mu}$ is constant on $X \times X'$.

Proof. Any point $x_0 \times x'_0$ has some $g = (g_1, g_2, g_3, g_4, g_5, g_6)$ such that $x_0 \times x'_0 = (g_1, g_2, g_3, g_4, g_5, g_6)x \times x'$. Then as $(g_1, g_2, g_3)v_\lambda$ is a highest weight element with respect to the subgroup given by x_0 (and a similar statement for $v_{-\lambda}$,

$$H_\lambda(x_0 \times x'_0) = \frac{\langle u_{-\lambda}, u_\lambda \rangle \langle (g_4, g_5, g_6)v_{-\lambda}, (g_1, g_2, g_3)v_\lambda \rangle}{\langle u_{-\lambda}, (g_1, g_2, g_3)v_\lambda \rangle \langle (g_4, g_5, g_6)v_{-\lambda}, u_\lambda \rangle} \quad (10)$$

But using the correspondence $v_\lambda \rightarrow f_K^\lambda, u_\lambda \rightarrow f_G^\lambda$, it's clear that each of the three products other than $\langle u_\lambda, u_{-\lambda} \rangle$ cancels; we therefore get that for any $x_0 \times x'_0$, we have

$$\frac{H_{\lambda+\mu}}{H_\lambda H_\mu} = \frac{\langle u_{-\mu-\lambda}, u_{\mu+\lambda} \rangle}{\langle u_{-\mu}, u_\mu \rangle \langle u_{-\lambda}, u_\lambda \rangle}$$

Therefore, it is constant. \square

This lemma shows that when we change our choice of subgroups, it changes $\frac{H(\lambda+\mu)}{H(\lambda)H(\mu)}$ by a constant independent of λ (and exponential in μ). As we only care about $b_\mu(\lambda)$ up to a constant (depending on μ , but not λ), this means that we can choose any subgroup to find the $H(\lambda)$.

It will turn out that we only need to know H on certain subcones of Ω . Ginzburg and Finkelberg found the generators of Ω in the proof of lemma 5.5.1 [2]; there are two families of generators. Label them as follows:

$$\alpha_i = (\wedge^i \omega, \wedge^{n-i} \omega, 0), 1 \leq i \leq n-1$$

$$\beta_j = (\wedge^{j-1} \omega, \wedge^{n-j} \omega, 1), 1 \leq j \leq n$$

We then only need find H on the following subcones:

$$\Delta = \text{span}(\{\alpha_i\}_{i=1}^n)$$

$$\Delta_{<j} = \text{span}(\beta_j \cup \{\alpha_i\}_{i < j})$$

$$\Delta_{\geq j} = \text{span}(\beta_j \cup \{\alpha_i\}_{i \geq j})$$

For each j , choose B_1, B'_2 as the upper triangular subgroup, B'_1, B_2 as the lower triangular subgroup, P fixing e_j , and P' fixing e'_j . Let R_+ be the set of positive roots of $G = SL_n$. For each j , define the following function on positive roots:

For $\gamma \in R_+$, define

$$\chi_j(\gamma) = \begin{cases} 1 & \text{if } \gamma > \wedge^j \omega \\ 1 & \text{if } \gamma > \wedge^{j-1} \omega \\ 0 & \text{else} \end{cases} \quad (11)$$

Theorem 13. If $\lambda \in \Delta$, then $H(\lambda) = \dim(V_{\lambda_1})$. If $\lambda \in \Delta_{i \geq j}$ or $\lambda \in \Delta_{i < j}$ with $\lambda = (\sum_i r_i \alpha_i) + s_j \beta_j$, then

$$H(\lambda) = \frac{\prod_{\gamma \in R_+} (h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi_j(\gamma))}{\prod_{\gamma \in R_+} h_\gamma(\rho)}$$

Proof. If $\lambda \in \Delta$, then $\lambda = (\lambda_1, \lambda_1^*, 0)$, where $V_{\lambda_1^*} = V_{\lambda_1}^*$. Then by choosing $u_\lambda \in V_\lambda$ corresponding to the identity map on V_{λ_1} , $u_{-\lambda} \in V_{-\lambda}^*$ corresponding to the trace map, $v_\lambda = v_{\lambda_1} \otimes v_{-\lambda_1}$ and $v_{-\lambda} = v_{-\lambda_1} \otimes v_{\lambda_1}$, we can see that:

$$H(\lambda) = \dim(V_{\lambda_1}) = \prod_{\gamma \in R_+} \frac{h_\gamma(\lambda_1 + \rho)}{h_\gamma(\rho)} \quad (12)$$

If $\lambda \in \Delta_{\geq j}$, then let

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, l) = s_j \beta_j + \sum_{i \geq j} r_i \alpha_i \\ \lambda' &= (\lambda'_1, \lambda'_2, l') = s_j \alpha_j + \sum_{i \geq j} r_i \alpha_i \\ \lambda^0 &= (\lambda_1^0, \lambda_2^0, l^0) = \sum_{i \geq j} r_i \alpha_i \end{aligned}$$

We now define several maps to be used later. Let us denote the wedge map $\wedge_j : V_{\wedge^{j-1}\omega} \otimes \mathbb{C}^n \rightarrow V_{\wedge^j\omega}$; note that it is G -equivariant. By copying this map s_j times, we get a map $\wedge_j^{s_j} : V_{s_j \wedge^{j-1}\omega} \otimes \text{Sym}^{s_j} \mathbb{C}^n \rightarrow V_{s_j \wedge^j\omega}$ taking $v_{s_j \wedge^{j-1}\omega} \otimes v_{s_j \omega} \rightarrow v_{s_j \wedge^j\omega}$. Then by tensoring with $\text{Id}_{V_{\lambda_0}}$, we have a map $f : (V_{s_j \wedge^{j-1}\omega} \otimes V_{\lambda_1^0}) \otimes V_{\lambda_2^0} \otimes V_{s_j \omega} \rightarrow (V_{s_j \wedge^j\omega} \otimes V_{\lambda_1^0}) \otimes V_{\lambda_2^0}$. Then let $g : V_\lambda \rightarrow V_{\lambda'}$ be defined by the composition:

$$V_\lambda \xrightarrow{i_{s_j \wedge^{j-1}\omega, \lambda_1^0}} (V_{s_j \wedge^{j-1}\omega} \otimes V_{\lambda_1^0}) \otimes V_{\lambda_2^0} \otimes V_{s_j \omega} \xrightarrow{f} (V_{s_j \wedge^j\omega} \otimes V_{\lambda_1^0}) \otimes V_{\lambda_2^0} \xrightarrow{p_{s_j \wedge^j\omega, \lambda_1^0}} V_{\lambda'}' \quad (13)$$

where i is the natural inclusion map that takes v_λ to $v_{s_j \wedge^{j-1}\omega} \otimes v_{\lambda_1^0}$ and p is the natural quotient map that takes $v_{s_j \wedge^j\omega} \otimes v_{\lambda_1^0}$ to $v_{\lambda'}'$. Then g is G -equivariant and $g(v_\lambda) = v_{\lambda'}'$.

Because g is G -equivariant, we have that $g(u_\lambda) = g(H(\lambda)v_\lambda^G) = H(\lambda)g(v_\lambda)^G = H(\lambda)v_{\lambda'}^G = \frac{H(\lambda)}{H(\lambda')}u_{\lambda'}$. Therefore, if we define c by $g(u_\lambda) = cu_{\lambda'}$, then $H(\lambda) = cH(\lambda')$; as $\lambda' \in \Delta$, we know $H(\lambda')$, so we only need to find c .

Let $g^* : V_{\lambda'}^* \rightarrow V_\lambda^*$ be the transpose of g . By the definition of g , we have

$$g^*(v_{-\lambda'}) = g^*(v_{-\lambda'_1} \otimes v_{-\lambda'_2}) = (p \circ f)^*((v_{-\lambda_1^0} \otimes v_{-s_j \wedge^j\omega}) \otimes v_{-\lambda_2})$$

Using the definition of f , we want to know $(\wedge_j^{s_j})^* v_{-s_j \wedge^j\omega}$, where $(\wedge_j^{s_j})^*$ is the transpose of $\wedge_j^{s_j}$. Let $SL_j \subset G$ act on the first j coordinates in both \mathbb{C}^n and

its dual. By restricting our focus to the subspaces where SL_j acts nontrivially (and doing the same for the dual spaces), we obtain a SL_j -equivariant map $\wedge_j^{s_j} : V_{s_j \wedge^{j-1} \omega_j} \otimes V_{s_j \omega_j} \rightarrow V_{s_j \wedge^j \omega_j} = \mathbb{C}$ taking $v_{s_j \wedge^{j-1} \omega_j} \otimes v_{s_j \omega_j} \rightarrow 1$; in other words, a SL_j -invariant element $w \in V_{s_j \wedge^{j-1} \omega_j}^* \otimes V_{s_j \omega_j}^*$ such that $\langle w, v_{s_j \wedge^{j-1} \omega_j} \otimes v_{s_j \omega_j} \rangle = 1$.

But this corresponds to the conditions we've already studied if we take $G' = SL_j$, $\delta = (s_j \wedge^{j-1} \omega_j, s_j \omega_j, 0) \in \Delta_{G'}$; by definition, $w = u_{-\delta}$. Then by the uniqueness up to scalar of the G' -invariant, $w = c_1 v_{-\delta}^{G'}$ for some c_1 . By pairing with v_δ , we get that $1 = \langle w, v_\delta \rangle = \langle c_1 v_{-\delta}^{G'}, v_\delta \rangle = c_1 \langle v_{-\delta}, v_\delta^{G'} \rangle$. Then as $u_\delta = H_{SL_j}(\lambda) v_\delta^{G'}$, we get that $H_{SL_j}(\delta) = c_1 \langle v_{-\delta}, u_\delta \rangle = c_1$.

We therefore have that $w = H_{SL_j}(\delta) v_{-\delta}^{G'}$. By expanding our focus again to all n coordinates, we have that $\wedge_j^{s_j^*} v_{-s_j \wedge^j \omega} = H_{SL_j}(\delta) (v_{-s_j \wedge^{j-1} \omega} \otimes v_{-s_j \omega})^{SL_j}$. Therefore, as $v_{-\lambda_1^0}$ and $v_{-\lambda_2}$ are SL_j -invariant, $g^*(v_{-\lambda'}) = H_{SL_j}(\delta) v_{-\lambda}^{SL_j}$. Then by pairing with u_λ , we get that:

$$\begin{aligned} c &= \langle v_{-\lambda'}, cu_{\lambda'} \rangle = \langle v_{-\lambda'}, g(u_\lambda) \rangle = \langle g^*(v_{-\lambda'}), u_\lambda \rangle \\ &= \langle H_{SL_j}(\delta) v_{-\lambda}^{SL_j}, u_\lambda \rangle = H_{SL_j}(\delta) \langle v_{-\lambda}, u_\lambda \rangle = H_{SL_j}(\delta) \end{aligned}$$

where we use the G -invariance of u to say that $\langle v_{-\lambda}^{SL_j}, u_\lambda \rangle = \langle v_{-\lambda}, u_\lambda \rangle$.

We therefore have that $H(\lambda) = H_{SL_j}(\delta) H(\lambda')$. By the calculation of $H(\lambda)$ on Δ (equation 12), we get that:

$$\begin{aligned} H(\lambda') &= \frac{\prod_{\gamma \in R_+} (h_\gamma(\rho + (\sum_i r_i \wedge^i \omega) + s_j \wedge^j \omega))}{\prod_{\gamma \in R_+} (h_\gamma(\rho))} \\ H_{SL_j}(\delta) &= \frac{\prod_{\gamma \in R_{SL_j+}} (h_\gamma(\rho_{SL_j} + s_j \wedge^{j-1} \omega_j))}{\prod_{\gamma \in R_{SL_j+}} h_\gamma(\rho_{SL_j})} \end{aligned}$$

By considering R_{SL_j+} as a subset of R_+ , we get the second half of the theorem. \square

For $\gamma \in R_+$, define

$$\chi'_j(\gamma) = \begin{cases} 1 & \text{if } \gamma > \wedge^j \omega + \wedge^{j-1} \omega \\ 0 & \text{else} \end{cases} \quad (14)$$

Combining this calculation of $H(\lambda)$ and $b_{K,\mu}(\lambda)$ from earlier, we get:

Corollary 14. *We can find $b_{G^3,\mu}(\lambda)$ if there is a subcone listed above that both μ, λ are in. Specifically:*

If $\mu = \alpha_j, \lambda \in \Delta$ (so $\lambda = (\lambda_1, \lambda_1^, 0)$), then*

$$b_\mu(\lambda) = \frac{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho + \lambda_1) + 1)(h_\gamma(\rho + \lambda))}{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho) + 1)(h_\gamma(\rho))}$$

If $\mu = \beta_j, \lambda \in \Delta_{i < j}$ or $\lambda \in \Delta_{i \geq j}$ with $\lambda = (\sum_i r_i \alpha_i) + s_j \beta_j$, then

$$b_\mu(\lambda) = \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{and } \gamma > \wedge^j \omega}} \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j \right) \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{or } \gamma > \wedge^j \omega}} \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j + 1 \right)$$

If $\mu = \alpha_k, k < j, \lambda \in \Delta_{i < j}$ or $k \geq j, \lambda \in \Delta_{i \geq j}$, then

$$b_\mu(\lambda) = \prod_{\gamma > \wedge^k \omega} \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi'_j(\gamma) \right) \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi_j(\gamma) + 1 \right)$$

8 The b-function

We have now assembled enough rules to fully determine $b_{G^3, \mu}(\lambda)$ for any $\mu \in \Omega, \lambda \in \mathbb{C}\Omega$. By equation 5, it is enough to find $b_{G^3, \mu}$ for μ generators of Ω . Write $\lambda = \sum a_i \alpha_i + \sum b_j \beta_j$. We therefore want to find a set of polynomials $\{b_{\alpha_i}\}_{1 \leq i \leq n-1}, \{b_{\beta_j}\}_{0 \leq j \leq n-1}$ satisfying the following properties:

1. Each of the polynomials factors as a product of hyperplanes with integer coefficients, as in lemma 6.
2. If $a_i = -1$, $f^{\lambda + \alpha_i}$ is a G -invariant global section of a line bundle which has no nontrivial G -invariant global sections, so the functional equation implies that $b_{\alpha_i}(\lambda) = 0$, and therefore that $(a_i + 1) | b_{\alpha_i}$; this is analogous to the fact that $s + 1 | b(s)$. Similarly, $(b_j + 1) | b_{\beta_j}$.
3. If $(\delta + k) | b_{\alpha_i}$ for $\delta \in \text{Hom}(\Lambda, \mathbb{Z})$, then $\langle c\delta, \alpha_i \rangle \neq 0$, and similarly for b_{β_j} , as in corollary 8.
4. If $(\delta + k) | b_{\alpha_i}$ for $\delta \in \text{Hom}(\Lambda, \mathbb{Z})$, and $\langle \delta, \alpha_{i'} \rangle \neq 0$, then for some k' , $(\delta + k') | b_{\alpha_{i'}}$, and similarly for b_{β_j} , as in corollary 7.
5. If $(\delta + k) | b_{\alpha_i}$ for $\delta \in \text{Hom}(\Lambda, \mathbb{Z})$ and $\langle \delta, \alpha_i \rangle = \langle \delta, \alpha_j \rangle$, then $(\delta + k) | b_{\alpha_j}$, and similar results with b_{β_j} , as in corollary 9.
6. If $b_j = 0$ for all j , we have

$$b_{\alpha_i} = \frac{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho + \lambda_1) + 1)(h_\gamma(\rho + \lambda))}{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho) + 1)(h_\gamma(\rho))}.$$

If $s_j = 0$ for all $j \neq k$ and either $a_i = 0$ for either all $i < k$ or all $i \geq k$, we have

$$b_{\alpha_i} = \prod_{\gamma > \wedge^k \omega} \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi'_j(\gamma) \right) \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi_j(\gamma) + 1 \right)$$

$$b_{\beta_k} = \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{and } \gamma > \wedge^j \omega}} \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j \right) \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{or } \gamma > \wedge^j \omega}} \left(h_\gamma(\rho + \sum r_i \alpha_i) + s_j + 1 \right).$$

These will determine the b-functions entirely. Let Γ be the gamma function.

Theorem 15. Write $\lambda = (\sum_i a_i \alpha_i) + (\sum_j b_j \beta_j)$. Then let

$$A(\lambda) = \prod_j \Gamma(b_j + 1) \cdot \prod_{\gamma \in R_+} \Gamma \left(h_\gamma(\rho) + \sum_{\omega^i < \gamma} a_i + \sum_{\substack{\omega^{j-1} < \gamma \\ \text{and } \omega^j < \gamma}} b_j \right) \\ \cdot \Gamma \left(h_\gamma(\rho) + \sum_{\omega^j < \gamma} a_i + \sum_{\substack{\omega^{j-1} < \gamma \\ \text{or } \omega^j < \gamma}} b_j + 1 \right)$$

$$\text{Then } b_\mu(\lambda) = \frac{A(\lambda + \mu)}{A(\lambda)A(\mu)}.$$

Proof. Let $\alpha_i = (\wedge^i \omega, \wedge^{n-i} \omega, 0) \in \Delta$. By property 1, $b_{\alpha_i}(\lambda) = \prod_k \langle \eta_k, \lambda \rangle + m_k$

for some η_k, m_k such that $\langle \eta_k, \alpha_i \rangle \neq 0$. By property 6, we have that $b_{\alpha_i}(\lambda) = \frac{\prod_{\gamma \in R_+} (h_\gamma(\rho + \lambda_1) + 1) h_\gamma(\rho + \lambda)}{\prod_{\gamma \in R_+} (h_\gamma(\rho) + 1) h_\gamma(\rho)}$ if $\lambda \in \mathbb{C}\Delta$. By combining these, we get that

we can partition the factors of b_{α_i} into two families of hyperplanes. The families are both indexed by the roots $\gamma \in R_+$ such that $\gamma > \wedge^i \omega$; define L_{γ, α_i} as the hyperplane that, on Δ , is equal to $h_\gamma(\lambda_1) + h_\gamma(\rho) + 1$, and L'_{γ, α_i} as the hyperplane that, on Δ , is equal to $h_\gamma(\lambda_1) + h_\gamma(\rho)$.

By property 5, if $\gamma > \wedge^i \omega, \wedge^{i'} \omega$, then $L_{\gamma, \alpha_i} | b_{\alpha_{i'}}$; the only hyperplane it could match is $L_{\gamma, \alpha_{i'}}$, so $L_{\gamma, \alpha_i} = L_{\gamma, \alpha_{i'}}$. Similarly, $L'_{\gamma, \alpha_i} = L'_{\gamma, \alpha_{i'}}$. Thus, we can ignore the subscripts of α and just denote the hyperplanes as L_γ, L'_γ .

Choose some j . Then for each γ , there is some i such that $\gamma > \wedge^i \omega$. If $i < j$, then α_i and β_j are both in $\Delta_{<j}$, while if $i \geq j$, then they are both in $\Delta_{\geq j}$. In either case, by property 6, the coefficient of b_j in L_γ and L'_γ must either be 0 or equal to the coefficient of a_i . As the coefficients of all of the a_i are either 0 or 1, this implies that the coefficients of all of the b_j are either 0 or 1. By properties 3, 4, and 5, this implies that any hyperplane factor of b_{β_j} is either L_γ or L'_γ for some γ , or that its coefficients of a_i are all 0.

Let S_j , resp. S'_j be the set of γ such that the coefficient of b_j in L_γ , resp. L'_γ is 1. Let $S_{i,j}$, resp. $S'_{i,j}$ be the subset of S_j , resp. S'_j such that $\gamma > \wedge^i \omega$. As for each γ there is some i such that $\gamma > \wedge^i \omega$, $S_j = \cup S_{i,j}$, $S'_j = \cup S'_{i,j}$.

Assume without loss of generality that $i < j$, and therefore that α_i, β_j are in $\Delta_{i < j}$. Then by property 6, $|S_{i,j}| + |S'_{i,j}| = i(n-j) + i(n-j-1)$. Also by property 6, if $\gamma \in S_{i,j}$, then $h_\gamma(\rho) + 1 \geq i-j+1$, while if $\gamma \in S'_{i,j}$, then $h_\gamma(\rho) \geq i-j+1$. Finally, also by property 6, if $\gamma \in S_{i,j}$ or $S'_{i,j}$, then $\gamma > \wedge^{j-1} \omega$.

Then $\gamma \in S_{i,j}$ or $S'_{i,j}$ implies that $\gamma > \sum_{k=i}^{j-1} \wedge^k \omega$.

Assume $\sum_{k=i}^{j-1} \wedge^k \omega \in S'_j$ for some i . Then it is also in $S'_{i,j}$; therefore, $\gamma(\rho) \geq i - j + 1$. But $\gamma(\rho) = i - j$, so this is impossible - so no root of the form $\sum_{k=i}^{j-1} \wedge^k \omega \in S'_j$. Therefore, if $\gamma \in S'_{i,j}$, then $\gamma \geq \wedge^j \omega$.

We therefore have that if $\gamma \in S_{i,j}$, then $\gamma > \wedge^{j-1} \omega$, while if $\gamma \in S'_{i,j}$, then $\gamma > \wedge^j \omega$. But then $|S_{i,j}| \leq i(n-j)$, while $|S'_{i,j}| \leq i(n-j-1)$. We know that $|S_{i,j}| + |S'_{i,j}| = i(n-j) + i(n-j-1)$; therefore, we must have that all of the elements that could be in either must be, and therefore that $S_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^{j-1} \omega\}$, while $S'_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^j \omega\}$.

Correspondingly, if $i \geq j$ instead, then $S_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^j \omega\}$ while $S'_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^{j-1} \omega\}$. Then as $S_j = \cup S_{i,j}$, $S'_j = \cup S'_{i,j}$, we get that

$$S_j = \{\gamma | \gamma > \wedge^{j-1} \omega \text{ or } \gamma > \wedge^j \omega\}$$

and

$$S'_j = \{\gamma | \gamma > \wedge^{j-1} \omega \text{ and } \gamma > \wedge^j \omega\}$$

Therefore

$$\begin{aligned} L_\gamma &= \sum_{\gamma > \wedge^i \omega} a_i + \sum_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{or } \gamma > \wedge^j \omega}} b_j + h_\gamma(\rho) + 1 \\ L'_\gamma &= \sum_{\gamma > \wedge^i \omega} a_i + \sum_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{and } \gamma > \wedge^j \omega}} b_j + h_\gamma(\rho) \end{aligned}$$

This gives b_{α_i} as the product of the L_γ, L'_γ . We now only need to find b_{β_j} . By property 6, the degree of b_{β_j} is

$$(j-1)(n-j) + j(n-j-1) + 1 = (j-1)(n-j-1) + (j(n-j)-1) + 1$$

. But $|S'_j| = (j-1)(n-j-1)$, and $|S_j| = j(n-j)-1$, so only one factor is unaccounted for - and that factor, by property 2, is $b_j + 1$. By inspection, then, we get the formula in the theorem. \square

9 Appendix

Theorem 16. *Let V be a finite dimensional \mathfrak{b} -module. Then $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ is a $U\mathfrak{h} = S\mathfrak{h}$ module of rank equal to the dimension of the μ weight subspace of V .*

Proof. As V is finite-dimensional, we can define an increasing filtration by \mathfrak{b} -submodules F_i on V^* such that $F_{-1} = \{0\}$, F_{i+1}/F_i has a unique weight λ_i (note that F_{i+1}/F_i is not necessarily 1-dimensional), and the λ_i are distinct.

We have an isomorphism $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu} \simeq Hom_{\mathfrak{b},-\mu}(V^*, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$ where the subscript denotes that the homomorphism twists by $-\mu$; in other words, if $w \in W$ of weight λ and $f \in Hom_{\mathfrak{b},-\mu}(W, U)$, then $f(w)$ has weight $\lambda - \mu$. This homomorphism set has a residual right $U\mathfrak{h}$ action from its action on $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$. We proceed by induction on the filtration.

We can start with the base case of F_{-1} , the trivial \mathfrak{b} -module; by definition, $Hom_{\mathfrak{b},-\mu}(F_{-1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) = \{0\}$.

Assume $\lambda_{i+1} \neq \mu$. Because F_i is a submodule of F_{i+1} , we get a map $\iota_i : Hom_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) \rightarrow Hom_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$. Kashiwara [1] proved (equation (1.4)) that $\exists g_i \neq 0 \in S\mathfrak{h}$ such that $g_i Hom_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) \in Im(\iota_i)$.

Therefore $rk Hom_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) \geq rk Hom_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$ as $U\mathfrak{h}$ -modules.

Let $f \in Hom_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$ with $\iota_i(f) = 0$. Then $f(F_i) = 0$, so f descends to a map $\bar{f} : F_{i+1}/F_i \rightarrow \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ which twists by $-\mu$. For any $v \in F_{i+1}/F_i$, $nv = 0$, so $n\bar{f}(v) = 0$. But the only elements of $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ with this property are elements of $U\mathfrak{h}$, that is, elements of weight 0. As \bar{f} twists by $-\mu$, and $\lambda_{i+1} \neq \mu$, this is impossible unless $\bar{f} = 0$, and therefore $f = 0$ - so ι is injective. Therefore, we get that the ranks must in fact be equal.

Assume $\lambda_{i+1} = \mu$. As the λ_i are distinct, by the above, $Hom_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$ is trivial. The map ι_i is not injective in this case; we need to find its kernel. Choose a basis of $v_j \in F_{i+1}/F_i$; then for each j , we can define $\bar{f}_j : F_{i+1}/F_i \rightarrow \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ with $\bar{f}_j(v_k) = 1$ if $j = k, 0$ else. By inspection, these twist by $-\mu$. We can therefore extend these to $f \in Hom_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$, so $rk Hom_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) = dim F_{i+1}/F_i = dim V^\mu$. \square

Corollary 17. Assume $V = V_\nu^*$ is the dual of a finite-dimensional irreducible representation of $U\mathfrak{g}$ with highest weight ν . If $\nu < \mu$ or $\nu - \mu$ is not integral, then $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ is trivial.

Theorem 18. Assume that the dimension of V^μ is 1. Then $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$

is free as a right $U\mathfrak{h}$ -module.

In order to prove this, we will need a lemma about when a submodule of a free submodule is free. Let A be a UFD, and let N be an A -module. Define $A^{-1}N = \text{Frac}(A) \otimes_A N$.

Lemma 19. *Let A be a UFD, M a free A -module, and N a finitely generated submodule of M of rank 1. If $A^{-1}N \cap M = N$, then N is free.*

Proof. Note that $A^{-1}N \cap M = \{m \in M \mid \exists a \text{ s.t. } am \in N\}$. Let $\{n_i\}$ be a minimal generating set of N . Assume it contains at least two elements n, n' . If we can find some $n'' \in n$ that generates both n, n' , then by induction, we can show that N is free.

Because N is rank 1, there must be some b, b' relatively prime with $bn = b'n'$. Let $\{v_j\}$ be a basis for M ; then $n = \sum a_j v_j, n' = \sum a'_j v_j$. As they are a basis, we get that $ba_j = b'a'_j$. Then as A is a UFD, $b' \mid a_j$. Let $a''_j = \frac{a_j}{b'}$, and let $n'' = \sum a''_j v_j = \frac{n}{b'} = \frac{n'}{b}$. As $a''_j \in A$ for each j , $n'' \in M$, so $n'' \in A^{-1}N \cap M = N$. Therefore, the induction step is done, and N is free. \square

Proof of Theorem 18. Let $N = (V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$; by the first theorem, is a rank 1 $U\mathfrak{h}$ -module. As $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ is a free right $U\mathfrak{h}$ -module, so is $M = V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$. We then only need to prove that $N(U\mathfrak{h})^{-1} \cap M = N$.

By definition, $N(U\mathfrak{h})^{-1} \cap M = \{m \in M \mid \exists h \in U\mathfrak{h} \text{ } mh \in N\}$. Let $m \in M, h \in U\mathfrak{h}$ such that $mh \in N$. We need to prove two things: that for any $n \in \mathfrak{n}, nm = 0$, and that for any $h' \in \mathfrak{h}, h'm = m(h' - \mu)$. If $mh \in N$, then $nmh = 0$. But M is free as a right $U\mathfrak{h}$ -module, so $nm = 0$. If $mh \in N$, then $h'mh = mh(h' - \mu) = m(h' - \mu)h$. Again as M is free, $h'm = m(h' - \mu)$. Therefore, by the lemma, N is a free rank 1 $U\mathfrak{h}$ -module. \square

References

- [1] Kashiwara, M. *The Universal Verma Module and the b-function*, Algebraic Groups and Related Topics (Kyoto-Nagoya 1983), North-Holland, 1985, pp. 67-82
- [2] Finkelberg, M. and Victor Ginzburg, *On Mirabolic D-Modules*, arXiv:0803.0578v3 [math.AG]
- [3] Etingof, P. and Victor Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, arXiv:math/0011114 [math.AG]
- [4] Sabbah, C., *Proximite Evanescent II*, Compositio Math., 64 (1987), pp. 213-241

- [5] Gyoja, A., *Bernstein-Sato's polynomial for several analytic functions*, J. Math Kyoto University, 33-2 (1993), pp. 399-411